

Eigenvalue and Eigenvector Derivatives of a Nondefective Matrix

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A novel approach is introduced to address the problem of the existence of differentiable eigenvectors for a nondefective matrix that may have repeated eigenvalues. The existence of eigenvector derivatives for a unique set of continuous eigenvectors corresponding to a repeated eigenvalue is rigorously established for nondefective and analytic matrices. A numerically implementable method is then developed to compute the differentiable eigenvectors associated with repeated eigenvalues. The solutions of eigenvalue and eigenvector derivatives for repeated eigenvalues are then derived. An example is given to illustrate the validity of formulations developed in this paper.

I. Introduction

CURRENT space structures may be dynamically very complex. Situations frequently arise in which vibration modes are very close together in frequency. This condition may occur, for example, when structures are nearly symmetric in a plane or when loosely coupled branch systems have natural frequencies that are very close to those of the global system. Prediction of the dynamic behavior of structures under these conditions is difficult. The phasing of branch responses relative to the parent structure may be entirely reversed in certain modes, depending on the accuracy of a crucial stiffness parameter.

System performance or control stability may depend on the ability to predict the structural behavior under these sensitive conditions. Thus, understanding the sensitivity of the eigenvalues and eigenvectors with respect to some parameter such as a stiffness or mass value under conditions of close or repeated eigenvalues is essential. The eigenvalue and eigenvector derivatives for a matrix with *distinct* eigenvalues are well documented in Refs. 1–4. The general expressions for eigenvalue and eigenvector derivatives for non-self-adjoint systems appear to be given first in Ref. 1. The formulation was correct, but incomplete in the sense that only directional changes were included in the eigenvector sensitivity. Reference 4 reexamined the eigenvector derivatives by including the contribution of eigenvector change in the nominal eigenvector direction. The problem of eigenvalue derivatives and the continuity of eigenvectors for *repeated* eigenvalues has been addressed in Ref. 6 for the class of matrices that are nondefective and analytic. The existence of eigenvector derivatives corresponding to a repeated eigenvalue, however, has not been addressed. Assuming existence, a novel approach has been developed in Ref. 7 using the singular value decomposition technique to derive a numerically stable algorithm for computing eigenvector derivatives associated with repeated eigenvalues, but limited

to the case of distinct eigenvalue derivatives. Although Ref. 8 has tried to address the existence issue and proposed a computational method for nondefective matrices, the results were inconclusive in the sense that they were applicable only to very limited cases.

A novel approach is developed in this paper to prove the existence of the eigenvector derivatives associated with repeated eigenvalues for nondefective and analytic matrices. It is shown that a unique set of differentiable eigenvectors corresponding to a repeated eigenvalue can be identified. Along similar lines as in Ref. 7, numerically implementable formulations are derived to determine the differentiable eigenvectors and compute the eigenvector derivatives associated with repeated eigenvalues. In contrast to the approach in Ref. 7 that uses the singular value decomposition technique, a conventional approach, namely the modal expansion technique, is used to derive the formulations that include the case where repeated eigenvalues and repeated first eigenvalue derivatives occur. In addition, the approach developed in this paper can be extended easily to cover general cases in which repeated eigenvalues and repeated eigenvalue derivatives of a higher order than the first derivative occur. A simple example is presented to illustrate the concepts.

II. Existence of Eigenvector Derivatives

Consider the right eigenvalue problem for a nondefective matrix A of order n , whose elements are an analytic function of a real scalar parameter ρ

$$A(\rho)\psi_i(\rho) = \lambda_i(\rho)\psi_i(\rho) \quad (i = 1, \dots, n)$$

or

$$A(\rho)\Psi(\rho) = \Psi(\rho)\Lambda(\rho) \quad (1)$$

and the left eigenvalue problem

$$\phi_i^T(\rho)A(\rho) = \lambda_i(\rho)\phi_i^T(\rho) \quad (i = 1, \dots, n)$$

or

$$\Phi^T(\rho)A(\rho) = \Lambda(\rho)\Phi^T(\rho) \quad (2)$$

where

$$\Phi(\rho) = [\phi_1(\rho), \dots, \phi_n(\rho)]$$

$$\Psi(\rho) = [\psi_1(\rho), \dots, \psi_n(\rho)]$$

$$\Lambda(\rho) = \text{diag} [\lambda_1(\rho), \dots, \lambda_n(\rho)]$$

For a nondefective matrix, there exists a full linearly independent set of eigenvectors. Moreover, the right and the left

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eigenvectors may be chosen to form a biorthonormal pair such that

$$\phi_i^T \psi_j = \delta_{ij}, \quad i, j = 1, \dots, n \quad (3)$$

Let $\lambda_1(\rho_0)$ be an eigenvalue of $A(\rho)$ at $\rho = \rho_0$ with multiplicity r_1 , that is, $\lambda_k(\rho_0) = \lambda_1(\rho_0)$ for $k = 1, 2, \dots, r_1$. Given that the eigenvectors of the matrix A are continuous,⁶ let the specific sets of right eigenvectors $\tilde{\Psi} = [\tilde{\psi}_1, \dots, \tilde{\psi}_{r_1}]$ and left eigenvectors $\tilde{\Phi} = [\tilde{\phi}_1, \dots, \tilde{\phi}_{r_1}]$ associated with $\lambda_1(\rho_0)$ be obtained through a continuation of the corresponding eigenspaces from an arbitrary point ρ to point ρ_0 . The objective of this section is to prove that every vector in the sets $\tilde{\Psi}$ and $\tilde{\Phi}$ is differentiable with respect to the parameter ρ .

Now consider the sets of vectors $\tilde{\Psi} = [\tilde{\psi}_1, \dots, \tilde{\psi}_{r_1}]$ and $\tilde{\Phi} = [\tilde{\phi}_1, \dots, \tilde{\phi}_{r_1}]$ as the bases for the right and the left eigenspace of $A(\rho)$, respectively. Any right eigenvector $\psi(\rho)$ or left eigenvector $\phi(\rho)$ can then be expressed as

$$\psi(\rho) = \sum_{i=1}^{r_1} \tilde{\psi}_i \alpha_i(\rho) \quad \text{and} \quad \phi(\rho) = \sum_{i=1}^{r_1} \tilde{\phi}_i \beta_i(\rho) \quad (4)$$

where $\alpha_i(\rho)$ and $\beta_i(\rho)$ are the appropriate coefficients of the basis vectors. The coefficients $\alpha_i(\rho)$ and $\beta_i(\rho)$, $i = 1, \dots, r_1$ are continuous since the eigenvectors of A are continuous.⁶

Let $\tilde{\psi}_1, \dots, \tilde{\psi}_{r_1}$ be the r_1 right eigenvectors corresponding to the eigenvalue $\lambda_1(\rho_0)$ with a multiplicity of r_1 . To show that a typical eigenvector, for example, $\tilde{\psi}_1$, is differentiable, consider the right eigenvalue problem at a perturbed point ρ

$$A(\rho)\psi(\rho) = \lambda(\rho)\psi(\rho) \quad (5)$$

such that $\lambda(\rho) \rightarrow \lambda_1(\rho_0)$ and $\psi(\rho) \rightarrow \tilde{\psi}_1$ as $\rho \rightarrow \rho_0$. Suppose that $A^{s_1}(\rho_0) = [\partial A^{s_1}(\rho)/\partial \rho^{s_1}]_{\rho=\rho_0}$ is the first nonvanishing derivative of $A(\rho)$ at ρ_0 . Then, if we use Taylor's theorem, the matrix $A(\rho)$ may be expanded as

$$A(\rho) = A(\rho_0) + (\rho - \rho_0)^{s_1} A_1(\rho, \rho_0) \quad (6)$$

where $A_1(\rho, \rho_0)$ is an analytic function, and

$$\lim_{\rho \rightarrow \rho_0} A_1(\rho, \rho_0) = A^{s_1}(\rho_0)/s_1!$$

For simplicity without losing generality, only the case where $s_1 = 1$ is addressed in the following development for the existence of eigenvector derivatives. Only slight modifications in formulations are required to cover the case where $s_1 > 1$.

A. Distinct Eigenvalue Derivatives

Using Eqs. (4) and (6) in Eq. (5) and premultiplying the resulting equation by $\tilde{\phi}_j^T$ for $r_1 + 1 \leq j \leq n$ yields

$$[\lambda(\rho_0) - \lambda_j(\rho)]\alpha_j(\rho) = (\rho - \rho_0)\tilde{\phi}_j^T A_1(\rho, \rho_0)\psi(\rho) \quad (7)$$

where $\lambda(\rho) \neq \lambda_j(\rho_0)$ for all ρ in the neighborhood of ρ_0 . Since

$$\lim_{\rho \rightarrow \rho_0} \psi(\rho) = \tilde{\psi}_1$$

it is clear that

$$\lim_{\rho \rightarrow \rho_0} \alpha_k(\rho) = 0$$

for $1 < k \leq n$ and

$$\lim_{\rho \rightarrow \rho_0} \alpha_1(\rho) = 1$$

Equation (7) produces

$$\begin{aligned} & \lim_{\rho \rightarrow \rho_0} \{[\alpha_j(\rho) - \alpha_j(\rho_0)]/(\rho - \rho_0)\} \\ &= \lim_{\rho \rightarrow \rho_0} \{ \tilde{\phi}_j^T A_1(\rho, \rho_0)\psi(\rho)/[\lambda_j(\rho) - \lambda(\rho_0)] \} \end{aligned} \quad (8)$$

for $r_1 + 1 \leq j \leq n$. The limit on the right side of Eq. (8) exists,

since $A_1(\rho, \rho_0)$ and $\psi(\rho)$ are continuous functions and $\lambda(\rho) \neq \lambda_j(\rho_0)$ for all ρ in the neighborhood of ρ_0 . Thus, the coefficients $\alpha_j(\rho)$ for $r_1 + 1 \leq j \leq n$ are differentiable with respect to the parameter ρ at $\rho = \rho_0$.

The differentiability of the remaining coefficients $\alpha_j(\rho)$ for $1 \leq j \leq r_1$ can be determined as follows. Substituting Eqs. (4) and (6) into Eq. (5) and premultiplying the resulting equation by $\tilde{\phi}_j^T$ for $1 < j \leq r_1$ yields

$$\begin{aligned} & \{[\lambda(\rho) - \lambda_1(\rho_0)]/(\rho - \rho_0) - \tilde{\phi}_j^T A_1(\rho, \rho_0)\tilde{\psi}_j\} \alpha_j(\rho) \\ &= \sum_{\substack{k=1 \\ k \neq j}}^{r_1} \tilde{\phi}_j^T A_1(\rho, \rho_0)\tilde{\psi}_k \alpha_k(\rho) \end{aligned} \quad (9)$$

Let the matrix $A_1(\rho, \rho_0)$ be expanded as

$$A_1(\rho, \rho_0) = A_1(\rho_0, \rho_0) + (\rho - \rho_0)^{s_2} A_2(\rho, \rho_0) \quad (10)$$

in which $A_2(\rho, \rho_0)$ is a matrix with analytic coefficients and s_2 is a positive integer. For simplicity without losing generality, only the case where $s_2 = 1$ is addressed. It is shown⁶ that

$$\tilde{\phi}_j^T A_1(\rho_0, \rho_0)\tilde{\psi}_k = \lambda_j'(\rho_0)\delta_{jk} \quad (11)$$

for $1 \leq j, k \leq r_1$ where $\lambda_j'(\rho_0)$ is the first derivative of $\lambda_j(\rho)$ at $\rho = \rho_0$. Substitution of Eq. (10) into Eq. (9) with the aid of Eq. (11) yields

$$\begin{aligned} \alpha_j(\rho) = & \left[\sum_{k > r_1}^n \tilde{\phi}_j^T A_1(\rho_0, \rho_0)\tilde{\psi}_k \alpha_k(\rho) \right. \\ & \left. + (\rho - \rho_0) \sum_{\substack{k=1 \\ k \neq j}}^{r_1} \tilde{\phi}_j^T A_2(\rho, \rho_0)\tilde{\psi}_k \alpha_k(\rho) \right] / [\lambda_j(\rho) - \lambda_j'(\rho)] \end{aligned} \quad (12)$$

for $1 < j \leq r_1$, where $\lambda_j'(\rho) \triangleq [\lambda(\rho) - \lambda_1(\rho_0)]/(\rho - \rho_0)$ and $\lambda_j'(\rho) \triangleq \tilde{\phi}_j^T A_1(\rho, \rho_0)\tilde{\psi}_j$. Note that

$$\partial \lambda_1 / \partial \rho|_{\rho=\rho_0} = \lim_{\rho \rightarrow \rho_0} \lambda_j'(\rho) = \lambda_j'(\rho_0)$$

and

$$\lambda_j'(\rho_0) = \partial \lambda_j / \partial \rho|_{\rho=\rho_0}$$

(see Ref. 6). If the first derivatives of $\lambda_j(\rho)$ at $\rho = \rho_0$ for $1 < j \leq r_1$ are distinct from those of $\lambda_1(\rho)$ at $\rho = \rho_0$, then the denominator $[\lambda_j(\rho) - \lambda_j'(\rho)]$ in Eq. (12) is nonzero as $\rho \rightarrow \rho_0$. With

$$\lim_{\rho \rightarrow \rho_0} \alpha_j(\rho) = 0$$

for $1 < j \leq n$ and

$$\lim_{\rho \rightarrow \rho_0} \alpha_1(\rho) = 1$$

Equation (12) produces

$$\begin{aligned} & \lim_{\rho \rightarrow \rho_0} \{[\alpha_j(\rho) - \alpha_j(\rho_0)]/(\rho - \rho_0)\} \\ &= \lim_{\rho \rightarrow \rho_0} \left(\sum_{k > r_1}^n \tilde{\phi}_j^T A_1(\rho_0, \rho_0)\tilde{\psi}_k \alpha_k(\rho) \right. \\ & \quad \left. / \{(\rho - \rho_0)[\lambda_j(\rho) - \lambda_j'(\rho)]\} \right) \\ & \quad + \lim_{\rho \rightarrow \rho_0} \{ \tilde{\phi}_j^T A_2(\rho, \rho_0) \sum_{\substack{k=1 \\ k \neq j}}^{r_1} \tilde{\psi}_k \alpha_k(\rho) / [\lambda_j(\rho) - \lambda_j'(\rho)] \} \end{aligned} \quad (13)$$

Since coefficients $\alpha_k(\rho)$ for $r_1 + 1 \leq k \leq n$ are differentiable at $\rho = \rho_0$ [see Eq. (8)] and the term $\tilde{\phi}_j^T A_2(\rho, \rho_0) \sum_{k=1, k \neq j}^{r_1} \tilde{\psi}_k \alpha_k(\rho)$ is continuous in the neighborhood of ρ_0 , the coefficients $\alpha_j(\rho)$, $1 < j \leq r_1$ are thus differentiable at $\rho = \rho_0$.

We have shown that the derivatives of all the components $\alpha_j(\rho)$, $j = 2, \dots, n$ except the component $\alpha_1(\rho)$ for the continuous vector $\tilde{\psi}_1$ exist at $\rho = \rho_0$. The existence of the derivative of the remaining component $\alpha_1(\rho)$ can be established easily through the normalization constraint imposed on the eigenvector, that is, $\psi(\rho)^T \psi(\rho) = 1$ for all ρ in the neighborhood of ρ_0 . Thus, the differentiability of the eigenvector $\tilde{\psi}_1$ is established with all the coefficients $\alpha_j(\rho)$, $j = 1, \dots, n$ shown to be differentiable at $\rho = \rho_0$. For the case in which all the eigenvalue derivatives are distinct, similar procedures can be used to demonstrate the existence of derivatives for the other right eigenvectors $\tilde{\psi}_k$, $k = 2, \dots, n$ as well as the left eigenvectors $\tilde{\phi}_k$, $k = 1, \dots, n$.

It is concluded that the continuous eigenvectors associated with a repeated eigenvalue but with nonrepeated eigenvalue derivatives are differentiable for a nondefective matrix whose elements are an analytic function of a real parameter.

B. Repeated Eigenvalue Derivatives

The existence of eigenvector derivatives also can be established for those coefficients for which the denominator $[\lambda'_i(\rho) - \lambda'_j(\rho)]$ in Eq. (12) goes to zero as $\rho \rightarrow \rho_0$. Let the matrix $A_2(\rho, \rho_0)$ be expanded as

$$A_2(\rho, \rho_0) = A_2(\rho_0, \rho_0) + (\rho - \rho_0)^{s_3} A_3(\rho, \rho_0) \quad (14)$$

in which $A_3(\rho, \rho_0)$ is a matrix with analytic coefficients and s_3 is a positive integer. For simplicity, without losing generality, only the case where $s_3 = 1$ is addressed. Substituting the expression for $\alpha_k(\rho)$, $r_1 + 1 \leq k \leq n$ from Eq. (7) into Eq. (12) gives

$$\alpha_j(\rho) = [\lambda_j^{\#}(\rho) + (\rho - \rho_0) \tilde{\phi}_j^T A_3(\rho, \rho_0) \sum_{i=1}^n \tilde{\psi}_i \alpha_i(\rho)] / [\lambda'_j(\rho) - \lambda'_j(\rho)] \quad (15)$$

where

$$\begin{aligned} \lambda_j^{\#}(\rho) &= \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] [\tilde{\phi}_k^T A_1(\rho, \rho_0) \sum_{i=1}^n \tilde{\psi}_i \alpha_i(\rho)] \\ &\quad / [\lambda(\rho) - \lambda_k(\rho_0)] + \sum_{i=1}^n \tilde{\phi}_j^T A_2(\rho_0, \rho_0) \tilde{\psi}_i \alpha_i(\rho) \\ \lambda_j''(\rho) &= \left\{ \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] [\tilde{\phi}_k^T A_1(\rho, \rho_0) \tilde{\psi}_j] \right. \\ &\quad \left. / [\lambda(\rho) - \lambda_k(\rho_0)] \right\} + \tilde{\phi}_j^T A_2(\rho_0, \rho_0) \tilde{\psi}_j \end{aligned}$$

and

$$\lambda_1''(\rho) = \left(\{ [\lambda(\rho) - \lambda_1(\rho_0)] / (\rho - \rho_0) \} - \lambda'_1(\rho_0) \right) / (\rho - \rho_0)$$

Note that

$$\frac{1}{2} \frac{\partial \lambda_1^2 / \partial \rho^2}{\rho = \rho_0} = \lim_{\rho \rightarrow \rho_0} \lambda_1''(\rho) = \lambda_1''(\rho_0)$$

and

$$\lambda_j''(\rho_0) = \frac{1}{2} \frac{\partial \lambda_j^2 / \partial \rho^2}{\rho = \rho_0}$$

(see Ref. 6).

Here, it is assumed that there are $r_2(\leq r_1)$ repeated first derivatives for the repeated eigenvalues $\lambda_1(\rho) = \dots = \lambda_{r_2}(\rho)$ at ρ_0 , that is, $\lambda'_1(\rho_0) = \dots = \lambda'_{r_2}(\rho_0)$. Let the term $\lambda_j^{\#}(\rho)$ for $j = 1, \dots, r_2$ be decomposed as follows:

$$\lambda_j^{\#}(\rho) = \lambda_j^{\#}(\rho) + \lambda_j^{\#}(\rho) + \lambda_j^{\#}(\rho) \quad (16)$$

where

$$\begin{aligned} \lambda_{j10}^{\#}(\rho) &= \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] [\tilde{\phi}_k^T A_1(\rho_0, \rho_0) \sum_{i=1}^{r_2} \tilde{\psi}_i \alpha_i(\rho)] \\ &\quad / [\lambda(\rho) - \lambda_k(\rho_0)] + \sum_{i=1}^{r_2} \tilde{\phi}_j^T A_2(\rho_0, \rho_0) \tilde{\psi}_i \alpha_i(\rho) \\ \lambda_{j11}^{\#}(\rho) &= (\rho - \rho_0) \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] \\ &\quad \times [\tilde{\phi}_k^T A_2(\rho_0, \rho_0) \sum_{i=1}^{r_2} \tilde{\psi}_i \alpha_i(\rho)] / [\lambda(\rho) - \lambda_k(\rho_0)] \\ \lambda_{j12}^{\#}(\rho) &= \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] [\tilde{\phi}_k^T A_1(\rho, \rho_0) \sum_{i=r_2+1}^n \tilde{\psi}_i \alpha_i(\rho)] \\ &\quad / [\lambda(\rho) - \lambda_k(\rho_0)] + \sum_{k>r_2}^n \tilde{\phi}_j^T A_2(\rho_0, \rho_0) \tilde{\psi}_k \alpha_k(\rho) \end{aligned}$$

For continuous coefficients $\alpha_i(\rho)$ ($i = 1, \dots, n$), the term $\lambda_{j11}^{\#}(\rho)$ is differentiable. On the other hand, for differentiable coefficients $\alpha_k(\rho)$ ($r_2 < k \leq n$) [see Eqs. (8) and (13)], the term $\lambda_{j12}^{\#}(\rho)$ is differentiable. As a result, if the term $\lambda_{j10}^{\#}(\rho)$ in Eq. (16) is differentiable, then $\lambda_j^{\#}(\rho)$ is differentiable. Indeed, the differentiability of $\lambda_{j10}^{\#}(\rho)$ is proven in the sequel.

Since $\alpha_j(\rho) \rightarrow 0$ as $\rho \rightarrow \rho_0$, it is required that the term $\lambda_{j10}^{\#}(\rho)$ goes to zero as $\rho \rightarrow \rho_0$, such that

$$\begin{aligned} \lambda_{j10}^{\#}(\rho_0) &= \lim_{\rho \rightarrow \rho_0} \lambda_{j10}^{\#}(\rho) \\ &= \left\{ \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] [\tilde{\phi}_k^T A_1(\rho_0, \rho_0) \tilde{\psi}_1] \right. \\ &\quad \left. / [\lambda(\rho) - \lambda_k(\rho_0)] \right\} + \tilde{\phi}_j^T A_2(\rho_0, \rho_0) \tilde{\psi}_1 = 0 \end{aligned} \quad (17)$$

This equation holds for the continuous vector $\tilde{\psi}_1$. If we follow the same procedures for the derivatives of other continuous eigenvectors $\tilde{\psi}_i$ for $i = 2, \dots, r_2$, corresponding to the repeated eigenvalue $\lambda_1(\rho_0)$ that have equal first eigenvalue derivatives, a similar equation can be derived such that

$$\begin{aligned} \lambda_{j10}^{\#}(\rho_0) &= \lim_{\rho \rightarrow \rho_0} \lambda_{j10}^{\#}(\rho) \\ &= \left\{ \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] [\tilde{\phi}_k^T A_1(\rho_0, \rho_0) \tilde{\psi}_i] \right. \\ &\quad \left. / [\lambda_1(\rho) - \lambda_k(\rho_0)] \right\} + \tilde{\phi}_j^T A_2(\rho_0, \rho_0) \tilde{\psi}_i = 0 \end{aligned} \quad (18)$$

for $i = 2, \dots, r_2$ and $i \neq j$. Observation of Eqs. (16)–(18) thus yields

$$\begin{aligned} \lambda_{j10}^{\#}(\rho) &= \lambda_{j10}^{\#}(\rho) - \sum_{i=1}^{r_2} \lambda_{j10}^{\#}(\rho_0) \alpha_i(\rho) \\ &= -[\lambda(\rho) - \lambda_1(\rho_0)] \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] [\tilde{\phi}_k^T A_1(\rho_0, \rho_0) \\ &\quad \sum_{i=1}^{r_2} \tilde{\psi}_i \alpha_i(\rho)] / \{ [\lambda(\rho) - \lambda_k(\rho_0)] [\lambda_k(\rho_0) - \lambda_1(\rho_0)] \} \end{aligned} \quad (19)$$

which, in turn, produces

$$\begin{aligned} \lim_{\rho \rightarrow \rho_0} [\lambda_{j10}^{\#}(\rho) / (\rho - \rho_0)] &= -\lim_{\rho \rightarrow \rho_0} \{ [\lambda(\rho) - \lambda_1(\rho_0)] / (\rho - \rho_0) \} \\ &\quad \sum_{k>r_1}^n [\tilde{\phi}_j^T A_1(\rho_0, \rho_0) \tilde{\psi}_k] [\tilde{\phi}_k^T A_1(\rho_0, \rho_0) \sum_{i=1}^{r_2} \tilde{\psi}_i \alpha_i(\rho)] \\ &\quad / \{ [\lambda(\rho) - \lambda_k(\rho_0)] [\lambda_k(\rho_0) - \lambda_1(\rho_0)] \} \end{aligned} \quad (20)$$

Since

$$\lim_{\rho \rightarrow \rho_0} \{[\lambda(\rho) - \lambda_1(\rho_0)]/(\rho - \rho_0)\} = \lambda'_1(\rho_0)$$

exists and $\alpha_i(\rho)$ for $i = 1, \dots, n$ are continuous, $\lambda_{10}^{\#}(\rho)$ is thus differentiable at $\rho = \rho_0$. We have proven that the term $\lambda_j^{\#}(\rho)$ in Eq. (15) is differentiable at $\rho = \rho_0$. If the second eigenvalue derivatives of $\lambda_j(\rho)$ at $\rho = \rho_0$ are distinct, then the denominator of Eq. (15) is nonzero as $\rho \rightarrow \rho_0$. As a result, the coefficients $\alpha_i(\rho)$, $1 < i \leq r_2$ are differentiable at $\rho = \rho_0$. Having shown that the derivatives of all the components $\alpha_j(\rho)$, $j = 2, \dots, n$ exist at $\rho = \rho_0$, we can establish the existence of the component $\alpha_1(\rho)$ through the normalization constraint imposed on the eigenvectors, that is, $\psi^T(\rho)\psi(\rho) = 1$ for all ρ in the neighborhood of ρ_0 . Thus, the differentiability of the eigenvectors $\tilde{\psi}_1$ is established with all the coefficients shown to be differentiable at $\rho = \rho_0$. A similar procedure may be followed to demonstrate the existence of derivatives for the remaining right eigenvectors $\tilde{\psi}_k$, $k = 2, \dots, r_2$, as well as the left eigenvectors $\tilde{\phi}_k$, $k = 1, \dots, r_2$.

We conclude that the continuous eigenvectors associated with repeated eigenvalue and repeated eigenvalue derivatives but with nonrepeated second eigenvalue derivatives are differentiable for a nondefective matrix whose elements are an analytic function of a real parameter.

For the case in which repeated eigenvalue derivatives of a higher but finite order occur, the same approach can be used to prove the existence of the derivatives for continuous eigenvectors. However, the formulations are expected to be very involved and complicated.

The sufficient conditions for the existence of eigenvector derivatives with repeated eigenvalues can be summarized as follows: 1) The matrix is analytic such that the corresponding eigenvectors are continuous; 2) The matrix is nondefective such that a full linearly independent set of eigenvectors exists; and 3) Distinct finite derivatives of the eigenvalues exist.

III. Formulations for Computation of Eigenvalue and Eigenvector Derivatives

We have shown that the eigenvalue and eigenvector derivatives exist for nondefective matrices that may have repeated eigenvalues. The objective of this section is to derive numerically implementable formulations to identify the differentiable eigenvectors and then compute the eigenvalue and eigenvector derivatives.

The first part of this section will determine the differentiable eigenvectors associated with a repeated eigenvalue. Take the partial derivative of Eq. (1) with respect to parameter ρ and premultiply the resulting equation by Φ^T

$$\Phi^T(\rho) \{ \partial[A(\rho) - \lambda_1(\rho)I] / \partial \rho \} \psi_1(\rho) + \Phi^T(\rho) [A(\rho) - \lambda_1(\rho)I] [\partial \psi_1(\rho) / \partial \rho] = 0 \quad (21)$$

If we assume that the eigenvalue λ_1 has a multiplicity r_1 , that is, $\lambda_k = \lambda_1$ for $k = 1, \dots, r_1$, then it is required from Eq. (21) that

$$\tilde{\Phi}_1^T A' \tilde{\psi}_k - \lambda'_k \tilde{\Phi}_1^T \tilde{\psi}_k = 0, \quad k = 1, \dots, r_1 \quad (22)$$

where

$$A' = \partial A(\rho) / \partial \rho|_{\rho=\rho_0}, \quad \lambda'_k = \partial \lambda_k(\rho) / \partial \rho|_{\rho=\rho_0}$$

$$\tilde{\Phi}_1 = [\tilde{\phi}_1, \dots, \tilde{\phi}_{r_1}] = \Phi_1(\rho_0) = [\phi_1(\rho_0), \dots, \phi_{r_1}(\rho_0)]$$

$$\tilde{\Psi}_1 = [\tilde{\psi}_1, \dots, \tilde{\psi}_{r_1}] = \Psi_1(\rho_0) = [\psi_1(\rho_0), \dots, \psi_{r_1}(\rho_0)]$$

Let

$$\hat{\Phi}_1 = [\hat{\phi}_1, \dots, \hat{\phi}_{r_1}] \quad \text{and} \quad \hat{\Psi}_1 = [\hat{\psi}_1, \dots, \hat{\psi}_{r_1}]$$

respectively, represent the collection of linearly independent left and right eigenvectors associated with the repeated eigenvalues λ_1 for the nondefective matrix $A(\rho_0)$. Note that the

eigenvectors in $\hat{\Phi}_1$ and $\hat{\Psi}_1$ generally are not the differentiable eigenvectors since the differentiable eigenvectors to be determined are unknown a priori. Since any linear combination of eigenvectors associated with a repeated eigenvalue is also an eigenvector, the column vectors of the matrices $\hat{\Phi}_1$ and $\hat{\Psi}_1$, respectively, span a left eigenspace $S_{\hat{\Phi}_1}$ and a right eigenspace $S_{\hat{\Psi}_1}$ for the eigenvalue λ_1 .

Since the differentiable vectors in $\hat{\Phi}_1$ and $\hat{\Psi}_1$ are, respectively, in the left eigenspace $S_{\hat{\Phi}_1}$ and the right eigenspace $S_{\hat{\Psi}_1}$, the matrix $\hat{\Phi}_1$ and the vector $\tilde{\psi}_k$ can then be expressed, respectively, as follows:

$$\tilde{\Phi}_1 = \hat{\Phi}_1 \Gamma_1 \quad \text{and} \quad \tilde{\psi}_k = \hat{\Psi}_1 \alpha_k \quad (k = 1, \dots, r_1) \quad (23)$$

where Γ_1 is a $r_1 \times r_1$ nonsingular matrix and α_k a $r_1 \times 1$ vector. Substitution of Eq. (23) into Eq. (22) thus yields

$$\Gamma_1^T [\hat{\Phi}_1^T A' \hat{\Psi}_1 - \lambda'_k \hat{\Phi}_1^T \hat{\Psi}_1] \alpha_k = 0 \quad (k = 1, \dots, r_1) \quad (24)$$

The square matrix Γ_1 and the vector α_k used in Eqs. (23) and (24) for the computation of differentiable eigenvectors lying in the eigenspace $S_{\hat{\Psi}_1}$ and $S_{\hat{\Phi}_1}$ change when different basis vectors are chosen in the matrices $\hat{\Psi}_1$ and $\hat{\Phi}_1$, respectively. However, the differentiable eigenvectors in $S_{\hat{\Psi}_1}$ and $S_{\hat{\Phi}_1}$ are independent of the choice of bases. The eigenvalue derivatives λ'_k also remain unchanged regardless of the changes of bases. Since the square matrix Γ_1 is nonsingular, Eq. (24) gives

$$[\hat{\Phi}_1^T A' \hat{\Psi}_1] \alpha_k = \lambda'_k [\hat{\Phi}_1^T \hat{\Psi}_1] \alpha_k \quad (25)$$

or

$$[\hat{\Phi}_1^T A' \hat{\Psi}_1] \alpha_k = \lambda'_k \alpha_k$$

if the chosen bases in biorthonormal, that is, $\hat{\Phi}_1^T \hat{\Psi}_1 = I_{r_1}$, where I_{r_1} is an identity matrix of order r_1 .

The differentiable eigenvectors that are determined by Eq. (25) can then be written as $\tilde{\psi}_k = \hat{\Psi}_1 \alpha_k$ for $k = 1, \dots, r_1$ with their corresponding eigenvalue derivatives λ'_k . Note that the value λ'_k is called the characteristic value of the pencil $[\hat{\Phi}_1^T A' \hat{\Psi}_1 - \lambda'_k \hat{\Phi}_1^T \hat{\Psi}_1]$ and α_k a corresponding left principal vector of the pencil.

A. Distinct Eigenvalue Derivatives

For distinct eigenvalue derivatives λ'_k ($k = 1, \dots, r_1$), the corresponding eigenvector derivatives are determined in the sequel. Partition the left and right eigenvector matrices Φ and Ψ for the nondefective matrix $A(\rho_0)$ as

$$\Phi = [\tilde{\Phi}_1 \quad \tilde{\Phi}_2] \quad \text{and} \quad \Psi = [\tilde{\Psi}_1 \quad \tilde{\Psi}_2] \quad (26)$$

where $\tilde{\Phi}_1$ and $\tilde{\Psi}_1$, defined earlier, are the left and right basis matrices of the r_1 -dimensional eigenspace $S_{\hat{\Phi}_1}$ and $S_{\hat{\Psi}_1}$, respectively. The $n \times (n - r_1)$ matrices $\tilde{\Phi}_2$ and $\tilde{\Psi}_2$ are matrices representing the complementary subspaces $S_{\hat{\Phi}_2}$ and $S_{\hat{\Psi}_2}$, respectively. The right eigenvector derivative $\tilde{\psi}_k$ then can be expressed as

$$\tilde{\psi}_k = \partial \psi_k(\rho) / \partial \rho|_{\rho=\rho_0} = [\tilde{\Psi}_1 \quad \tilde{\Psi}_2] \begin{Bmatrix} \bar{x}_k \\ \hat{x}_k \end{Bmatrix}, \quad k = 1, \dots, r_1 \quad (27)$$

where \bar{x}_k and \hat{x}_k are, respectively, the components of the eigenvector derivative $\tilde{\psi}_k$ in the eigenspaces $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$. Substitution of the differentiable eigenvector $\tilde{\psi}_k = \hat{\Psi}_1 \alpha_k$ and Eq. (27) into Eq. (21) $\rho = \rho_0$ yields

$$\tilde{\Phi}_1^T [A' - \lambda'_k I] \hat{\Psi}_1 \alpha_k + [\bar{\Lambda}_1 - \lambda'_k \bar{I}] \bar{x}_k = 0 \quad (28)$$

or

$$\bar{x}_k = -[\bar{\Lambda}_1 - \lambda'_k \bar{I}]^{-1} \tilde{\Phi}_1^T [A' - \lambda'_k I] \hat{\Psi}_1 \alpha_k$$

where $\bar{\Lambda}_1 = \text{diag} [\lambda'_{r_1+1}, \dots, \lambda'_n]$ is a $(n - r_1) \times (n - r_1)$ diagonal matrix and \bar{I} is an $(n - r_1) \times (n - r_1)$ identity matrix.

To determine the remaining coefficients \hat{x}_k for the differentiable eigenvector $\hat{\Psi}_1 \alpha_k$ in Eq. (27), take the second derivative of the right eigenvalue problem, Eq. (1), and premultiply by $\hat{\Phi}_1^T$

$$\hat{\Phi}_1^T [A'' - \lambda_k'' I] \hat{\Psi}_1 \alpha_k + 2 \hat{\Phi}_1^T [A' - \lambda_k' I] [\hat{\Psi}_1 \hat{x}_k + \hat{\Psi}_1 \hat{x}_k] = 0 \quad (29)$$

for $k = 1, \dots, r_1$, or

$$2 \hat{\Phi}_1^T [A' - \lambda_k' I] \hat{\Psi}_1 \hat{x}_k = -\hat{\Phi}_1^T [A'' - \lambda_k'' I] \hat{\Psi}_1 \alpha_k - 2 \hat{\Phi}_1^T [A' - \lambda_k' I] \hat{\Psi}_1 \hat{x}_k$$

where

$$A'' = \partial^2 A(\rho) / \partial \rho^2 |_{\rho=\rho_0}$$

and

$$\lambda_k'' = \partial^2 \lambda_k(\rho) / \partial \rho^2 |_{\rho=\rho_0}$$

Substitution of Eq. (28) for the explicit expression of \hat{x}_k yields

$$2 \hat{\Phi}_1^T [A' - \lambda_k' I] \hat{\Psi}_1 \hat{x}_k = \hat{\Phi}_1^T [\lambda_k'' I - A^{(2)}] \hat{\Psi}_1 \alpha_k \quad (30)$$

where

$$A^{(2)} = A'' - 2[A' - \lambda_k' I] \hat{\Psi}_1 [\bar{\Lambda}_1 - \lambda_k' I]^{-1} \hat{\Phi}_1^T [A' - \lambda_k' I]$$

The motivation behind taking the second derivative of the eigenvalue problem is that the first derivative equation does not provide any information on the coefficients \hat{x}_k associated with the repeated eigenvalue λ_1 . Therefore, the second derivative equation is considered to provide additional information.

Since the nonrepeated eigenvalue derivative λ_k' is the characteristic value of the pencil $[\hat{\Phi}_1^T A' \hat{\Psi}_1] - \lambda_k' [\hat{\Phi}_1^T \hat{\Psi}_1]$ as shown in Eq. (25), the pencil has rank $(r_1 - 1)$. Thus, Eq. (30) has only $r_1 - 1$ independent equations to compute $r_1 - 1$ elements of the $r_1 \times 1$ vector \hat{x}_k with one arbitrary element undetermined. Let the differentiable eigenvector be determined such that

$$\hat{\Psi}_k^T(\rho) \hat{\Psi}_k(\rho) = 1 \quad (31)$$

Taking a partial derivative of Eq. (31) with respect to the parameter ρ and substituting Eq. (27) into the resulting equation gives

$$\alpha_k^T \hat{\Psi}_1^T [\hat{\Psi}_1 \hat{x}_k + \hat{\Psi}_1 \hat{x}_k] = 0 \quad (32)$$

Equation (32) represents a normalization constraint whereby all eigenvectors are normalized to a unit length. A combination of Eqs. (30) and (32) will then determine all the n elements of the coefficient vector x_k of the eigenvector derivative.

Examination of Eq. (30) reveals that λ_k'' remains to be determined. Let the $r_1 \times 1$ vector β_k be the right principal vector of the pencil $[\hat{\Phi}_1^T A' \hat{\Psi}_1] - \lambda_k' [\hat{\Phi}_1^T \hat{\Psi}_1]$ such that

$$\beta_k^T \{ [\hat{\Phi}_1^T A' \hat{\Psi}_1] - \lambda_k' [\hat{\Phi}_1^T \hat{\Psi}_1] \} = 0 \quad (33)$$

Premultiplying Eq. (30) by the vector β_k^T along with Eq. (33) produces

$$\beta_k^T \hat{\Phi}_1^T [\lambda_k'' I - A^{(2)}] \hat{\Psi}_1 \alpha_k = 0 \quad (34)$$

which implies

$$\lambda_k'' = [\beta_k^T \hat{\Phi}_1^T \hat{\Psi}_1 \alpha_k]^{-1} \beta_k^T \hat{\Phi}_1^T A^{(2)} \hat{\Psi}_1 \alpha_k \quad (35)$$

or

$$\lambda_k'' = \beta_k^T \hat{\Phi}_1^T A^{(2)} \hat{\Psi}_1 \alpha_k$$

if β_k , α_k , $\hat{\Phi}_1$, and $\hat{\Psi}_1$ are normalized such that $\beta_k^T \alpha_k = 1$ and $\hat{\Phi}_1^T \hat{\Psi}_1 = I_1$.

B. Repeated Eigenvalue Derivatives

It is possible that some eigenvalue derivatives λ_k' from Eq. (25) are also repeated for the repeated eigenvalue $\lambda_k = \lambda_1$ of the matrix $A(\rho_0)$. In this case, a unique differentiable eigenvector cannot be identified since no unique solution for the left principal vector of the pencil associated with the repeated eigenvalue derivative λ_k' can be obtained from Eq. (25). Assume that the eigenvalue derivative λ_k' has a multiplicity r_2 , that is, $\lambda_k' = \lambda_1'$ for $k = 1, \dots, r_2$. Note that $r_2 \leq r_1$, where r_1 is the multiplicity of the repeated eigenvalue λ_1 . Let $\Omega_1 = [\beta_1, \dots, \beta_{r_2}]$ and $\Theta_1 = [\alpha_1, \dots, \alpha_{r_2}]$, respectively, represent the collection of the right and left principal vectors of the pencil given by Eq. (25), associated with the repeated eigenvalue derivatives λ_k' . Since any linear combination of the principal vectors is also a principal vector of the eigenvalue derivative, the column vectors of Ω_1 and Θ_1 , respectively, span a right eigenspace S_{β_1} and a left eigenspace S_{α_1} for the eigenvalue derivative λ_k' . Let the $r_1 \times r_2$ matrices $\hat{\Omega}_1$ and $\hat{\Theta}_1$, respectively, collect the basis vectors for the eigenspaces S_{β_1} and S_{α_1} . The $r_1 \times r_2$ matrix Ω_1 and the $r_1 \times 1$ vector α_k can then be expressed as follows:

$$\Omega_1 = \hat{\Omega}_1 \Gamma_1^{(1)} \quad \text{and} \quad \alpha_k = \hat{\Theta}_1 \alpha_k^{(1)} \quad (36)$$

where $\Gamma_1^{(1)}$ is a $r_2 \times r_2$ nonsingular matrix and $\alpha_k^{(1)}$ a $r_2 \times 1$ vector. Substituting Eq. (36) into Eq. (30) and premultiplying the resulting equation by $\hat{\Omega}_1^T$ yields

$$\hat{\Omega}_1^{(1)T} \hat{\Omega}_1^T \hat{\Phi}_1^T [A^{(2)} - \lambda_k'' I] \hat{\Psi}_1 \hat{\Theta}_1 \alpha_k^{(1)} = 0, \quad k = 1, \dots, r_2 \quad (37)$$

Although the $r_2 \times r_2$ matrix $\Gamma_1^{(1)}$ and the $r_2 \times 1$ vector $\alpha_k^{(1)}$ of Eq. (37) depend on the choice of the basis vectors represented by the columns of $\hat{\Omega}_1$ and $\hat{\Theta}_1$, the solutions to Eq. (37) that are used to compute the differentiable eigenvectors $\hat{\Psi}_k = \hat{\Psi}_1 \hat{\Theta}_1 \alpha_k^{(1)}$ for $k = 1, \dots, r_2$ in S_{β_1} are independent of the choice of basis vectors. The second derivative of the eigenvalues λ_k'' is also invariant to the change of basis. Since the square matrix $\hat{\Omega}_1^{(1)}$ is nonsingular, Eq. (37) gives

$$\hat{\Omega}_1^T \hat{\Phi}_1^T A^{(2)} \hat{\Psi}_1 \hat{\Theta}_1 \alpha_k^{(1)} = \lambda_k'' [\hat{\Omega}_1^T \hat{\Phi}_1^T \hat{\Psi}_1 \hat{\Theta}_1] \alpha_k^{(1)} \quad (38)$$

or

$$\hat{\Omega}_1^T \hat{\Phi}_1^T A^{(2)} \hat{\Psi}_1 \hat{\Theta}_1 \alpha_k^{(1)} = \lambda_k'' \alpha_k^{(1)}$$

if the normalizations $\hat{\Phi}_1^T \hat{\Psi}_1 = I_{r_1}$ and $\hat{\Omega}_1^T \hat{\Theta}_1 = I_{r_2}$ are imposed. The differentiable eigenvectors thus determined by the solution of the characteristic equation of Eq. (38) can be expressed as $\hat{\Psi}_k = \hat{\Psi}_1 \hat{\Theta}_1 \alpha_k^{(1)}$, for $k = 1, \dots, r_2$, with the corresponding second eigenvalue derivatives λ_k'' . For the case in which the characteristic values λ_k'' of the pencil, Eq. (38), are distinct, the corresponding eigenvector derivatives can be obtained through a procedure similar to the case with distinct eigenvalue derivatives.

Partition the coefficient vector x_k in Eq. (27) such that

$$\tilde{x}_k = \Psi x_k \triangleq \begin{bmatrix} \hat{\Psi}_1 & \hat{\Psi}_1 \hat{\Theta}_1 & \hat{\Psi}_1 \hat{\Theta}_1 \end{bmatrix} \begin{Bmatrix} \hat{x}_k \\ \hat{x}_k^{(1)} \\ \hat{x}_k^{(1)} \end{Bmatrix}, \quad k = 1, \dots, r_2 \quad (39)$$

and, correspondingly, $\Phi = [\hat{\Phi}_1 \quad \hat{\Phi}_1 \hat{\Omega}_1 \quad \hat{\Phi}_1 \hat{\Omega}_1]$. Note that $\hat{x}_k^{(1)}$ denotes the components for x_k that are associated with repeated eigenvalue derivatives λ_k' in the eigenspace of the repeated eigenvalue λ_1 . The $n \times (n - r_1)$ matrices $\hat{\Phi}_1$ and $\hat{\Psi}_1$ are defined in Eq. (26). The $n \times r_2$ matrix $\hat{\Psi}_1 \hat{\Theta}_1$ provides the differentiable eigenvectors determined by $\hat{\Psi}_k = \hat{\Psi}_1 \hat{\Theta}_1 \alpha_k^{(1)}$ ($k = 1, \dots, r_2$), whereas the $n \times (n - r_1 - r_2)$ matrix $\hat{\Psi}_1 \hat{\Phi}_1$ contains all the remaining eigenvectors associated with those eigenvalue derivatives different from λ_k' .

Substitution of the differentiable eigenvector $\tilde{\psi}_k = \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)}$ and Eq. (39) into Eq. (21) yields

$$\tilde{\Phi}_1^T [A' - \lambda_k I] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)} + \tilde{\Phi}_1^T [A - \lambda_k I] \tilde{\Psi}_1 \tilde{x}_k = 0 \quad (40)$$

or

$$\tilde{x}_k = -\{\tilde{\Phi}_1^T [A - \lambda_k I] \tilde{\Psi}_1\}^{-1} \tilde{\Phi}_1^T [A' - \lambda_k I] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)}$$

If we similarly substitute Eq. (39) and the vector $\tilde{\psi}_k = \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)}$ into the second derivative of the right eigenvalue problem, Eq. (1), at $\rho = \rho_0$ and premultiply the resulting equation by $\tilde{\Omega}_1^T \tilde{\Phi}_1^T$, an equation similar to Eq. (30) is obtained

$$2\tilde{\Omega}_1^T \tilde{\Phi}_1^T [A' - \lambda_k I] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)} = \tilde{\Omega}_1^T \tilde{\Phi}_1^T [\lambda_k'' I - A^{(2)}] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)} \quad (41)$$

or

$$2\tilde{x}_k^{(1)} = \{\tilde{\Omega}_1^T \tilde{\Phi}_1^T [A' - \lambda_k I] \tilde{\Psi}_1\}^{-1} \tilde{\Omega}_1^T \tilde{\Phi}_1^T [\lambda_k'' I - A^{(2)}] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)}$$

for $k = 1, \dots, r_2$. Let $\partial \psi_k^2 / \partial \rho^2|_{\rho=\rho_0}$ be expanded such that

$$\tilde{\psi}_k'' = \partial \psi_k^2 / \partial \rho^2|_{\rho=\rho_0} = [\tilde{\Psi}_1 \quad \tilde{\Psi}_1 \tilde{\Theta}_1 \quad \tilde{\Psi}_1 \tilde{\Theta}_1] \begin{Bmatrix} \tilde{y}_k \\ \tilde{y}_k^{(1)} \\ \tilde{y}_k^{(1)} \end{Bmatrix} \quad (42)$$

which is similar to Eq. (39) for the eigenvector derivative. Substituting this equation into the second derivative of the right eigenvalue problem, Eq. (1), and premultiplying by $\tilde{\Phi}_1^T$ yields

$$\tilde{\Phi}_1^T [A - \lambda_k I] \tilde{\Psi}_1 \tilde{y}_k = -\tilde{\Phi}_1^T \{[A'' - \lambda_k'' I] \tilde{\psi}_k + 2[A' - \lambda_k I] \tilde{\psi}_k'\} \quad (43)$$

or

$$\tilde{y}_k = -\{\tilde{\Phi}_1^T [A - \lambda_k I] \tilde{\Psi}_1\}^{-1} \tilde{\Phi}_1^T \{[A'' - \lambda_k'' I] \tilde{\psi}_k + 2[A' - \lambda_k I] \tilde{\psi}_k'\}$$

This is one of the key equations for determining the remaining coefficient vector $\tilde{x}_k^{(1)}$.

Now taking the third derivative of the right eigenvalue problem, Eq. (1), premultiplying the remaining equation by $\tilde{\Omega}_1^T \tilde{\Phi}_1^T$ with the aid of Eq. (42), and noting that $\tilde{\Omega}_1^T \tilde{\Phi}_1^T [A' - \lambda_k I] \tilde{\Psi}_1 = 0$ yields

$$\tilde{\Omega}_1^T \tilde{\Phi}_1^T \{[A''' - \lambda_k''' I] \tilde{\psi}_k + 3[A'' - \lambda_k'' I] \tilde{\psi}_k' + 3[A' - \lambda_k I] \tilde{\psi}_k''\} = 0 \quad (44)$$

where

$$A''' = \partial^3 A(\rho) / \partial \rho^3|_{\rho=\rho_0} \quad \text{and} \quad \lambda_k''' = \partial^3 \lambda_k(\rho) / \partial \rho^3|_{\rho=\rho_0}$$

Substitution of the explicit expression for the vector \tilde{y}_k , Eq. (43), thus yields

$$\tilde{\Omega}_1^T \tilde{\Phi}_1^T \{3[A^{(2)} - \lambda_k'' I] \tilde{\psi}_k + [\tilde{A}^{(3)} - \lambda_k'' I] \tilde{\psi}_k\} = 0 \quad (45)$$

where $A^{(2)}$ is defined in Eq. (30) and

$$\tilde{A}^{(3)} = A''' - 3[A' - \lambda_k I] \tilde{\Psi}_1 \{\tilde{\Phi}_1^T [A - \lambda_k I] \tilde{\Psi}_1\}^{-1} \tilde{\Phi}_1^T [A'' - \lambda_k'' I]$$

A combination of Eqs. (39-41) results in

$$3\tilde{\Omega}_1^T \tilde{\Phi}_1^T [A^{(2)} - \lambda_k'' I] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)} = -\tilde{\Omega}_1^T \tilde{\Phi}_1^T [\tilde{A}^{(3)} - \lambda_k'' I] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)} - 3\tilde{\Omega}_1^T \tilde{\Phi}_1^T [A^{(2)} - \lambda_k'' I] [\tilde{\Psi}_1 \tilde{x}_k + \tilde{\Psi}_1 \tilde{\Theta}_1 \tilde{x}_k^{(1)}] \quad (46)$$

or

$$3\tilde{\Omega}_1^T \tilde{\Phi}_1^T [A^{(2)} - \lambda_k'' I] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)} = -\tilde{\Omega}_1^T \tilde{\Phi}_1^T [\tilde{A}^{(3)} - \lambda_k'' I] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)}$$

where

$$\begin{aligned} A^{(3)} &= \tilde{A}^{(3)} - 3[A^{(2)} - \lambda_k'' I] (\tilde{\Psi}_1 \{\tilde{\Phi}_1^T \\ &\times [A - \lambda_k I] \tilde{\Psi}_1\}^{-1} \tilde{\Phi}_1^T [A' - \lambda_k I] \\ &+ 1/2 \tilde{\Psi}_1 \{\tilde{\Omega}_1^T \tilde{\Phi}_1^T [A' - \lambda_k I] \tilde{\Psi}_1 \tilde{\Theta}_1\}^{-1} \tilde{\Omega}_1^T \tilde{\Phi}_1^T [\lambda_k'' I - A^{(2)}]) \end{aligned}$$

Equation (46) provides $r_2 - 1$ constraint equations on the r_2 elements of $\tilde{x}_k^{(1)}$. An additional constraint equation can be imposed through the normalization procedure described by Eq. (32) to determine the unique eigenvector derivative.

Examination of Eq. (46) indicates that the third eigenvalue derivative is required for the computation of the coefficient vector $\tilde{x}_k^{(1)}$. To obtain this derivative, let the $r_2 \times 1$ vector $\beta_k^{(1)}$ be the right principal vector of the pencil $\tilde{\Omega}_1^T \tilde{\Phi}_1^T [A^{(2)} - \lambda_k'' I] \tilde{\Psi}_1 \tilde{\Theta}_1$ such that

$$\beta_k^{(1)T} \tilde{\Omega}_1^T \tilde{\Phi}_1^T [A^{(2)} - \lambda_k'' I] \tilde{\Psi}_1 \tilde{\Theta}_1 = 0 \quad (47)$$

Premultiplying Eq. (46) by the vector $\beta_k^{(1)T}$ produces

$$\beta_k^{(1)T} \tilde{\Omega}_1^T \tilde{\Phi}_1^T [A^{(3)} - \lambda_k'' I] \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)} = 0 \quad (48)$$

which implies

$$\lambda_k'' = [\beta_k^{(1)T} \tilde{\Omega}_1^T \tilde{\Phi}_1^T \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)}]^{-1} \beta_k^{(1)T} \tilde{\Omega}_1^T \tilde{\Phi}_1^T A^{(3)} \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)}$$

or

$$\lambda_k'' = \beta_k^{(1)T} \tilde{\Omega}_1^T \tilde{\Phi}_1^T A^{(3)} \tilde{\Psi}_1 \tilde{\Theta}_1 \alpha_k^{(1)} \quad (49)$$

if proper normalizations are imposed.

For the case in which eigenvalue derivatives of second order are still repeated, that is, no unique differentiable eigenvector can be identified, a parallel approach involving invariant subspaces or pencils of higher orders in eigenvalue derivatives needs to be considered. This procedure is continued until one of the higher derivatives of the eigenvalue becomes distinct.

IV. Summary of Computational Steps

The computational steps for eigenvector derivatives associated with repeated eigenvalues are summarized as follows:

1) Compute the eigenvalues and eigenvectors of the matrix $A(\rho_0)$ and determine the multiplicity of each eigenvalue, Eq. (1).

2) Solve the eigenvalue problem, Eq. (25), to determine the first eigenvalue derivatives and the differentiable eigenvectors associated with a repeated eigenvalue.

3) Determine the eigenvector derivatives using Eqs. (27), (28), (30), and (35) for the repeated eigenvalues with distinct eigenvalue derivatives.

4) Calculate the eigenvector derivatives using Eqs. (38-41), (46), and (49) for those repeated eigenvalues and repeated eigenvalue derivatives but limited to the distinct second eigenvalue derivatives.

V. Numerical Example

To illustrate the validity of the formulations derived in this paper, consider the following third-order problem where

$$A = [1/(\rho^3 + 3\rho^2 - 1)]$$

$$\times \begin{bmatrix} \rho^6 + 4\rho^3 + 5\rho^2 & -\rho^4 + \rho^3 + \rho & \rho^5 - \rho^4 - \rho^2 \\ -\rho - 2 & & \\ -2\rho^6 - 2\rho^5 - \rho^4 & \rho^6 + 3\rho^5 + 3\rho^4 + 3\rho^3 & \rho^6 + \rho^5 - 3\rho^3 \\ + \rho^3 - \rho & + 5\rho^2 - \rho - 1 & \\ 2\rho^6 - 2\rho^5 - 5\rho^4 & \rho^5 + \rho^4 + 4\rho^3 - \rho^2 & 3\rho^5 + 4\rho^3 + 2\rho^2 \\ -3\rho^3 + \rho & & -\rho - 1 \end{bmatrix} \quad (50)$$

The eigenvector derivatives of matrix $A(\rho)$ are desired at $\rho = \rho_0 = 0$.

Step 1

Compute the eigensolution of the matrix $A(0)$

$$A(0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (51)$$

which simply gives the eigenvalues $\lambda = 2, 1, 1$ and their corresponding right and left eigenvectors

$$\Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (52)$$

The differentiable eigenvector corresponding to the repeated eigenvalue ($\lambda = 1$) and its derivative is to be determined. Let $\tilde{\Psi}_1$, $\tilde{\Psi}_1$, $\tilde{\Phi}_1$, and $\tilde{\Phi}_1$ of Eq. (26) be chosen as

$$\tilde{\Psi}_1 = \tilde{\Phi}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{\Psi}_1 = \tilde{\Phi}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (53)$$

From Eq. (50)

$$A'(0) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad (54)$$

Step 2

Solve the eigenvalue problem of Eq. (25). The first eigenvalue derivatives corresponding to the repeated eigenvalues ($\lambda = 1$) are found to be $\lambda' = 1, 1$. Since the eigenvalue derivatives are also repeated, the differentiable eigenvector associated with the repeated eigenvalue $\lambda = 1$ cannot be identified from step 3 directly.

Step 4

Find the matrices $\hat{\Theta}_1$ and $\hat{\Omega}_1$ of Eq. (36) from the solution of the eigenvalue problem of Eq. (25), which are

$$\hat{\Theta}_1 = \hat{\Omega}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (55)$$

The second eigenvalue derivatives are determined by solving the eigenvalue problem of Eq. (38) that yields the results $\lambda'' = -2, 2$. Since the second eigenvalue derivatives associated with the repeated eigenvalues $\lambda = 1$ are distinct, one differentiable eigenvector is computed as $\tilde{\Psi}_1 = \tilde{\Psi}_1 \hat{\Theta}_1 \alpha_1^{(1)}$, yielding

$$\tilde{\Psi}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (56)$$

The x_1 component of the eigenvector derivative [see Eq. (39)] is obtained from Eq. (40) and is equal to 1. The $\tilde{x}_1^{(1)}$ component of the derivative does not exist since the multiplicity of the repeated eigenvalues and their eigenvalue derivatives is the same. The remaining $\tilde{x}_1^{(1)}$ component is computed from Eqs. (46) and (31), which yield

$$\tilde{x}_1^{(1)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad (57)$$

Note that the third eigenvalue derivative, $\lambda''' = 6$, required in Eq. (46) is obtained from Eq. (50). As a result, the derivative of the differentiable eigenvector, Eq. (56), is given as

$$\tilde{\Psi}'_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad (58)$$

Similar procedures are followed for the other differentiable eigenvector associated with the repeated eigenvalue.

Let the matrix $\tilde{\Psi}$ represent the collection of the differentiable eigenvectors and the matrix $\tilde{\Psi}'$ the corresponding derivatives. The matrix in Eq. (50) gives

$$\tilde{\Psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{\Psi}' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \quad (59)$$

where the first column is the differentiable vector associated with a distinct eigenvalue that can be easily computed by Eq. (25) (a scalar equation for this case).

Conclusions

The existence of differentiable eigenvalues and eigenvectors for nondefective and analytic matrices is addressed. The differentiable eigenvectors are determined and numerically computed by using the modal expansion approach. The derivatives for the differentiable eigenvectors associated with repeated eigenvalues are numerically computed using higher-order derivatives of the matrix and the highest order required is problem-dependent. Although this paper covers to the case in which second-order eigenvalue derivatives associated with repeated eigenvalues are distinct, the approach developed in this paper can be extended to the general case in which higher-order eigenvalue derivatives are repeated.

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